PROPERTIES AND SOLUTION OF TRIANGLE

INTRODUCTION

In any triangle ABC, the side BC, opposite to the angle A is denoted by a ; the side CA and AB, opposite to the angles B and C respectively are denoted by b and c respectively. Semiperimeter of the triangle is denoted by s and its area by Δ or S. In this chapter, we shall discuss various relations between the sides a , b, c and the angles A,B,C of Δ ABC.

SINE RULE

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Note :- (1) The above rule can also be written as $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

COSINE FORMULAE

In any Δ ABC, $\cos A=\frac{b^2+c^2-a^2}{2bc}$, $\cos B=\frac{c^2+a^2-b^2}{2ac}$, $\cos C=\frac{a^2+b^2-c^2}{2ab}$

PROJECTION FORMULAE

In any Δ ABC, a = b cos C + c cos B, b = c cos A + a cos C, c = a cos B + b cos A.

TRIGONOMETRICAL RATIOS OF HALF OF THE ANGLES OF A TRIANGLE

In any Δ ABC , we have

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ac}}$, $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$, $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$, $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

AREA OF A TRIANGLE

If Δ be the area of a triangle ABC, then

(i) $\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

(ii) $\Delta=\frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)}=\frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)}=\frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

(iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Hero's formula)

Form above results, we obtain following values of sin A, sin B, sin C

(iv) $\sin A = \frac{2\Delta}{bc} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

(v) $\sin B = \frac{2\Delta}{ca} = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(vi) $\sin C = \frac{2\Delta}{ab} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Further with the help of (iv) , (v) (vi) we obtain $\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}}$

NAPIER'S ANALOGY

In any triangle ABC, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

CIRCUMCIRCLE OF A TRIANGLE

A circle passing through the vertices of a triangle is called the circumcircle of the triangle.

The centre of the circumcircle is called the circum-centre of the triangle and it is the point of intersection of the perpendicular bisectors of the sides of the triangle.

The radius of the circumcircle is called the circum radius of the triangle and is usually denoted by R and is given by the following formulae

$$\boxed{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}}$$

Where Δ is area of triangle and $s = \frac{a+b+c}{2}$

INCIRCLE OF A TRIANGLE

The circle which can be inscribed within the triangle so as to touch all the three sides is called the incircle of the triangle.

The centre of the incircle is called the in centre of the triangle and it is the point of intersection of the internal bisectors of the angles of the triangle.

The radius of the incircle is called the inradius of the triangle and is usually denoted by r and is given by the following formula

In – Radius : The radius r of the inscribed circle of a triangle ABC is given by

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\left(\frac{A}{2}\right)$, $r = (s-b)\tan\left(\frac{B}{2}\right)$ and $r = (s-c)\tan\left(\frac{C}{2}\right)$

(iii) $r = \dfrac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$, $r = \dfrac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ and $r = \dfrac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r = 4R\ \sin\left(\frac{A}{2}\right).\ \sin\left(\frac{B}{2}\right).\ \sin\left(\frac{C}{2}\right)$

ESCRIBED CIRCLES OF A TRIANGLE

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the escirbed circle opposite to the angle A. Its radius is denoted by r_1 . Similarly r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively.

The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisector of angles B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_1.

FORMULAE FOR r_1 , r_2 , r_3

In any Δ ABC, we have

(i) $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$, $r_2 = s\ \tan\dfrac{B}{2}$, $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = a\dfrac{\cos\frac{B}{2}\ \cos\frac{C}{2}}{\cos\frac{A}{2}}$, $r_2 = b\dfrac{\cos\frac{C}{2}\ \cos\frac{A}{2}}{\cos\frac{B}{2}}$, $r_3 = c\dfrac{\cos\frac{A}{2}\ \cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$, $r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$, $r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

ORTHOCENTRE OF A TRIANGLE

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre.

Let the perpendicular AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of ABC, respectively, meet at O. Then O is the orthocentre of the Δ ABC.

The triangle DEF is called the pedal triangle of the Δ ABC.

The distances of the orthocentre from the vertices and the sides - If O is the orthocentre and DEF the pedal triangle of the DABC, where AD, BE, CF are the perpendiculars drawn from A, B,C on the opposite sides BC, CA, AB respectively, then

(i) OA = 2R cos A, OB = 2R cos B and OC = 2R cos C

(ii) OD = 2R cos B cos C, OE = 2R cos C cos A and OF = 2R cos A cos B

(iii) The circumradius of the pedal triangle = $\dfrac{R}{2}$

(iv) The area of pedal triangle = 2Δ cosA cos B cos C.

SOME IMPORTANT RESULTS

(1) $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{s-c}{s}$ $\quad\therefore\quad$ $\cot\frac{A}{2}\cot\frac{B}{2}=\frac{s}{s-c}$

(2) $\tan\frac{A}{2}+\tan\frac{B}{2}=\frac{c}{s}\cot\frac{C}{2}=\frac{c}{\Delta}(s-c)$

(3) $\tan\frac{A}{2}-\tan\frac{B}{2}=\frac{a-b}{\Delta}(s-c)$

(4) $\cot\frac{A}{2}+\cot\frac{B}{2}=\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{c}{s-c}\cot\frac{C}{2}$

(5) Also note the following identities

(i) $S(p-q)=(p-q)+(q-r)+(r-p)=0$

(ii) $S\,p(q-r)=p(q-r)+q(r-p)+r(p-q)=0$

(iii) $S(p+a)\;(q-r)\;\;Sp(q-r)+a\,S(q-r)=0$